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Symmetry techniques for the Al-Salam–Chihara polynomials

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Abstract. An algebraic interpretation for the Al-Salam–Chihara q-orthogonal polynomials is presented. Using an explicit realization in terms of divided difference operators, their symmetry q-algebra is determined and studied. This algebraic structure allows one to derive a new expansion formula for these polynomials.

1. Introduction

The use of algebraic methods in the study of basic or q-orthogonal polynomials [1, 2] has proved very useful. Many formulae and identities involving these polynomials are direct consequences of the 'symmetry' algebras they possess. Indeed, most families of q-orthogonal polynomials are found to be basis vectors for irreducible representations of suitable q-deformations of classical Lie algebras [3–11].

Among the families of basic polynomials, of particular interest are those that are orthogonal with respect to a continuous measure. For specific choices of the parameters, these are all suitable limits of the Askey–Wilson polynomials, the most general class of orthogonal polynomials that is known [2, 12].

So far, only a small number of these q-polynomials have been given an algebraic interpretation: they are q-generalizations of the classical Jacobi, ultraspherical and Hermite polynomials [8–11]. In this paper we present a symmetry interpretation for a family of continuous q-orthogonal polynomials that does not have a classical analogue: the so-called Al-Salam–Chihara polynomials [2]. As discussed in section 2, the symmetry algebra for these polynomials is found to be an inhomogeneous q-algebra.

As an example of the application of these symmetry techniques, in section 3 a new expansion formula for the Al-Salam–Chihara polynomials will be algebraically derived. It is similar to the classical Fourier–Gegenbauer expansion relation for the Jacobi polynomials [13].

The basic orthogonal polynomials can be expressed in terms of generalized hypergeometric functions [1]:

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q; z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} [(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n \qquad (1.1) $$

where

$$(a_1, a_2, \ldots, a_k; q)_\alpha = (a_1; q)_\alpha (a_2; q)_\alpha \ldots (a_k; q)_\alpha \tag{1.2a}$$

and

$$(a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty} \qquad (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k) \qquad |q| < 1. \tag{1.2b}$$

Indeed, from the definition of the q-shifted factorial $(a; q)_\alpha$, one immediately sees that the series ${}_r\phi_s$ terminates if one of the parameters a_i, $i = 1, \ldots, r$, is equal to q^{-n}, with n a positive integer.

The Al-Salam–Chihara polynomials $Q_n(x; a, b|q)$ are two-parameter orthogonal polynomials in the variable $x = \cos\theta$, given explicitly by [2]:

$$Q_n(x; a, b|q) = \frac{(ab; q)}{a^n} {}_3\phi_2\left(\begin{matrix} q^{-n},\ a\,\mathrm{e}^{\mathrm{i}\theta},\ a\,\mathrm{e}^{-\mathrm{i}\theta} \\ ab, \quad 0 \end{matrix} \middle| q; q \right) \tag{1.3a}$$

$$= \mathrm{e}^{\mathrm{i}n\theta} (b\,\mathrm{e}^{-\mathrm{i}\theta}; q)_n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, a\,\mathrm{e}^{\mathrm{i}\theta} \\ b^{-1} q^{1-n}\,\mathrm{e}^{\mathrm{i}\theta} \end{matrix} \middle| q; qb^{-1}\,\mathrm{e}^{-\mathrm{i}\theta} \right). \tag{1.3b}$$

They are a particular case of the four-parameter Askey–Wilson polynomials [12]:

$$p_n(x; a, b, c, d|q) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3\left(\begin{matrix} q^{-n},\ abcd\,q^{n-1},\ a\,\mathrm{e}^{\mathrm{i}\theta},\ a\,\mathrm{e}^{-\mathrm{i}\theta} \\ ab, \quad ac, \quad ad \end{matrix} \middle| q; q \right). \tag{1.4}$$

When the four parameters a, b, c, d are real, the Askey–Wilson polynomials are orthogonal over the interval $0 < \theta < \pi$ with respect to the continuous measure

$$w(\cos\theta; a, b, c, d) = \left| \frac{(\mathrm{e}^{2i\theta}; q)_\infty}{(a\,\mathrm{e}^{\mathrm{i}\theta}, b\,\mathrm{e}^{\mathrm{i}\theta}, c\,\mathrm{e}^{\mathrm{i}\theta}, d\,\mathrm{e}^{\mathrm{i}\theta}; q)_\infty} \right|^2. \tag{1.5}$$

The orthogonality of the Al-Salam–Chihara polynomials follows from this property.

Note that as $q \to 1^-$, the polynomials $Q_n(x; a, b|q)$ become the simple monomials $(2x - a - b)^n$. Other more interesting limits involve setting one or both parameters a and b to zero. In these cases the polynomials $Q_n(x; a, b|q)$ reduce to q-generalizations of the classical Hermite polynomials, the so-called continuous big q-Hermite polynomials $(b = 0)$:

$$H_n(x; a|q) = a^{-n} {}_3\phi_2\left(\begin{matrix} q^{-n},\ a\,\mathrm{e}^{\mathrm{i}\theta},\ a\,\mathrm{e}^{-\mathrm{i}\theta} \\ 0, \quad 0 \end{matrix} \middle| q; q \right) \tag{1.6}$$

and the continuous q-Hermite polynomials $(a = b = 0)$

$$H_n(x|q) = \mathrm{e}^{\mathrm{i}n\theta}\ {}_2\phi_0\left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix} \middle| q; q^n\,\mathrm{e}^{-2\,\mathrm{i}\theta} \right). \tag{1.7}$$

An algebraic interpretation for these q-orthogonal polynomials has been presented in [8–10]. In the following, we shall show that a similar interpretation also holds for the Al-Salam–Chihara polynomials.

2. Algebraic interpretation

The mathematical structure that is at the basis of the algebraic interpretation of Al-Salam–Chihara polynomials is a generalized Euclidean q-algebra $\mathcal{G}_q$. We shall see that these polynomials occur as basis vectors of an irreducible representation of $\mathcal{G}_q$. This algebraic model is realized in terms of difference operators acting on functions of the three variables t,

s and $x = \frac{1}{2}(z + 1/z)$, with $z = \mathrm{e}^{\mathrm{i}\theta}$. The building blocks are the following four q-difference operators [14]:

$$\tau = \frac{1}{z - z^{-1}}(T_z^{1/2} - T_z^{-1/2}) \tag{2.1a}$$

$$\tau^\star = \frac{q^{-1/2}}{z - z^{-1}}\left[\frac{1}{z^2}(1 - q^{-1/2} z T_t^{1/2})(1 - q^{-1/2} z T_s^{1/2}) T_z^{1/2} - z^2 \left(1 - \frac{q^{-1/2}}{z} T_t^{1/2}\right)\left(1 - \frac{q^{-1/2}}{z} T_s^{1/2}\right) T_z^{-1/2}\right] \tag{2.1b}$$

$$\mu = \frac{1}{z - z^{-1}}\left[-\frac{1}{z}(1 - q^{-1/2} z T_t^{1/2})(1 - q^{-1/2} z T_s^{1/2}) T_z^{1/2} + z \left(1 - \frac{q^{-1/2}}{z} T_t^{1/2}\right)\left(1 - \frac{q^{-1/2}}{z} T_s^{1/2}\right) T_z^{-1/2}\right] \tag{2.1c}$$

$$\mu^\star = \frac{1}{z - z^{-1}}\left(-\frac{1}{z} T_z^{1/2} + z T_z^{-1/2}\right). \tag{2.1d}$$

In writing these expressions use has been made of the q-dilatation operator T_w^α, acting as follows on any function $f(w)$ of the generic variable w: $T_w^\alpha f(w) = f(q^\alpha w)$, α being any real number.

Let us now consider the following generators:

$$\begin{aligned}
&A_+ = -\frac{ts}{1-q}\tau && B_+ = ts\,\mu^\star \\
&A_- = \frac{1}{ts}\tau^\star && B_- = \frac{1}{ts(1-q)}\mu \\
&K_1 = T_t^{1/2} && K_2 = T_s^{1/2} \\
&Q_1 = t^2 && Q_2 = s^2 \\
&P = 2x &&
\end{aligned}$$

$$R_1 = \frac{t}{s}\frac{1}{(z - z^{-1})}\left[-\frac{1}{z}(1 - q^{-1/2} z T_s^{1/2}) T_t^{1/2} T_z^{1/2} + z\left(1 - \frac{q^{-1/2}}{z} T_s^{1/2}\right) T_t^{1/2} T_z^{-1/2}\right]$$

$$R_2 = \frac{s}{t}\frac{1}{(z - z^{-1})}\left[-\frac{1}{z}(1 - q^{-1/2} z T_t^{1/2}) T_s^{1/2} T_z^{1/2} + z\left(1 - \frac{q^{-1/2}}{z} T_t^{1/2}\right) T_s^{1/2} T_z^{-1/2}\right]. \tag{2.2}$$

Note that the last two operators are not independent from the others. Indeed, one can check that the following two identities hold:

$$Q_2 R_1 = (B_+ - q^{1/2}(1-q) A_+ K_2) K_1 \tag{2.3a}$$

$$Q_1 R_2 = (B_+ - q^{1/2}(1-q) A_+ K_1) K_2. \tag{2.3b}$$

Nevertheless, they are of great help in writing down explicitly the relations characterizing the q-algebra $\mathcal{G}_q$ that the operators (2.2) generate. In fact, one finds

$$\begin{aligned}
&A_- A_+ - q A_+ A_- = 1 && [B_-, B_+] = q^{-1} K_1 K_2 \\
&K_1 A_\pm = q^{\pm 1/2} A_\pm K_1 && K_1 B_\pm = q^{\pm 1/2} B_\pm K_1
\end{aligned}$$

$$\begin{aligned}
&P A_+ - q^{-1/2} A_+ P = -q^{-1} B_+ && P A_- - q^{1/2} A_- P = -q^{-1/2}(1-q)^2 B_- \\
&P B_+ - q^{1/2} B_+ P = (1-q)^2 A_+ && P B_- - q^{-1/2} B_- P = q^{-1/2} A_- \\
&A_+ B_+ = q^{-1/2} B_+ A_+ && A_- B_- = q^{1/2} B_- A_- \\
&A_+ B_- = q^{-1/2} B_- A_+ && A_- B_+ = q^{1/2} B_+ A_-
\end{aligned}$$

$$
\begin{aligned}
&[A_-, Q_1] = -q^{-1}(1-q)R_1 && B_-Q_1 - qQ_1B_- = R_1K_1^{-1}\\
&A_\pm R_1 = q^{\mp 1}R_1A_\pm && B_\pm R_1 = q^{\mp 1/2}R_1B_\pm\\
&Q_1R_1 = q^{-1}R_1Q_1 && Q_1R_2 - q^{-1}R_2Q_1 = -q^{-1}(1-q)B_+K_2\\
&[A_+, Q_1] = 0 && [B_+, Q_1] = 0
\end{aligned}
$$

$$
\begin{aligned}
&R_1P - q^{1/2}PR_1 = (1-q)(R_1 - A_-Q_1)K_1 && [P, Q_1] = 0\\
&[R_1, R_2] = 0 && [Q_1, Q_2] = 0\\
&[K_1, K_2] = 0 && [K_1, P] = 0\\
&K_1Q_1 = qQ_1K_1 && [K_1, Q_2] = 0\\
&K_1R_1 = q^{1/2}R_1K_1 && K_1R_2 = q^{-1/2}R_2K_1
\end{aligned}
\tag{2.4}
$$

plus the relations that are obtained by letting $1 \leftrightarrow 2$.

This q-algebra contains various interesting q-subalgebras. The set of operators $\{A_+, A_-, K_1\}$ and $\{B_+, B_-, K_1K_2\}$ generate two different forms of the q-oscillator algebra. Furthermore, A_+, B_+ q-commute and are 'rotated' one into the other by P; these three generators form, therefore, a q-deformation of the two-dimensional Euclidean algebra. The same is true for the set $\{A_-, B_-, P\}$. Additional q-subalgebras involve the operators $\{A_-, Q_1, R_1\}$, $\{B_-, Q_1, R_1K_1^{-1}\}$ and $\{Q_1, R_2, B_+K_2\}$; they all constitute q-deformations of certain contractions of $sl(2)$.

Note that in the limit $q \to 1$, the algebra (2.4) greatly simplifies, since most of the operators trivialize. Explicitly, by writing $K_i = q^{J_i/2}$, $i = 1, 2$, one obtains:

$$
\begin{aligned}
&A_+ \to \frac{ts}{2}\partial_x && B_+ \to ts\\
&A_- \to \frac{2}{ts}(1-x) && B_- \to \frac{1}{ts}\left[(x-1)\partial_x + \frac{1}{2}(t\partial_t + s\partial_s) - 1\right]\\
&J_1 \to t\partial_t && J_2 \to s\partial_s\\
&R_1 \to t/s && R_2 \to s/t
\end{aligned}
\tag{2.5}
$$

while P, Q_1 and Q_2 remain unaffected. In the same limit, the non-vanishing commutators are

$$
\begin{aligned}
&[A_-, A_+] = 1 && [B_-, B_+] = 1\\
&[J_1, A_\pm] = \pm A_\pm && [J_1, B_\pm] = \pm B_\pm\\
&[P, A_+] = -B_+ && [P, B_-] = A_-\\
&[J_1, Q_1] = 2\,Q_1 && [B_-, Q_1] = R_1\\
&[J_1, R_1] = R_1 && [J_1, R_2] = -R_2
\end{aligned}
\tag{2.6}
$$

and the ones that are obtained from these with the exchange $1 \leftrightarrow 2$.

An irreducible representation for the algebra $\mathcal{G}_q$ in (2.4) can be obtained by letting the operators (2.2) act on the following basis functions:

$$
f_n^{(\alpha,\beta)}(x, t, s) = t^\alpha s^\beta Q_n(x; q^{\alpha/2}, q^{\beta/2}|q). \tag{2.7}
$$

One can check that the following relations hold:

$$
\begin{aligned}
&A_+ f_n^{(\alpha,\beta)} = q^{-n/2}\frac{(1-q^n)}{(1-q)} f_{n-1}^{(\alpha+1,\beta+1)} && A_- f_n^{(\alpha,\beta)} = -q^{-(n+1)/2} f_{n+1}^{(\alpha-1,\beta-1)}\\
&B_+ f_n^{(\alpha,\beta)} = q^{-n/2} f_n^{(\alpha+1,\beta+1)} && B_-\, f_n^{(\alpha,\beta)} = q^{-n/2}\frac{(1-q^{\alpha/2+\beta/2+n-1})}{(1-q)} f_n^{(\alpha-1,\beta-1)}
\end{aligned}
$$

$$
\begin{aligned}
&K_1 f_n^{(\alpha,\beta)} = q^{\alpha/2} f_n^{(\alpha,\beta)} \qquad\qquad K_2 f_n^{(\alpha,\beta)} = q^{\beta/2} f_n^{(\alpha,\beta)}\\
&P f_n^{(\alpha,\beta)} = f_{n+1}^{(\alpha,\beta)} + q^n(q^{\alpha/2}+q^{\beta/2}) f_n^{(\alpha,\beta)} + (1-q^n)(1-q^{\alpha/2+\beta/2+n-1}) f_{n-1}^{(\alpha,\beta)}\\
&Q_1 f_n^{(\alpha,\beta)} = f_n^{(\alpha+2,\beta)} - q^{\alpha/2}(1-q^n) f_{n-1}^{(\alpha+2,\beta)}\\
&Q_2 f_n^{(\alpha,\beta)} = f_n^{(\alpha,\beta+2)} - q^{\beta/2}(1-q^n) f_{n-1}^{(\alpha,\beta+2)}\\
&R_1 f_n^{(\alpha,\beta)} = q^{(\alpha-n)/2} f_n^{(\alpha+1,\beta-1)} \qquad\qquad R_2 f_n^{(\alpha,\beta)} = q^{(\beta-n)/2} f_n^{(\alpha-1,\beta+1)}.
\end{aligned}
\tag{2.8}
$$

Note that the action of P reproduces the three-term recurrence relation for the Al-Salam–Chihara polynomials $Q_n(x; a, b|q)$,

$$
\begin{aligned}
2xQ_n(x; a, b|q) = {}& Q_{n+1}(x; a, b|q) + q^n(a+b)Q_n(x; a, b|q)\\
&+(1-q^n)(1-abq^{n-1})Q_{n-1}(x; a, b|q)
\end{aligned}
\tag{2.9}
$$

while the action of the two operators Q_1 and Q_2 is a consequence of the following two identities:

$$Q_n(x; a, b|q) = Q_n(x; qa, b|q) - a(1-q^n)Q_{n-1}(x; qa, b|q) \tag{2.10a}$$

$$Q_n(x; a, b|q) = Q_n(x; a, qb|q) - b(1-q^n)Q_{n-1}(x; a, qb|q). \tag{2.10b}$$

These can be proved by noting that both sides of formulae (2.10) satisfy the recurrence relation (2.9), with the same initial conditions.

A simple computation shows that the relations (2.8) reproduce the algebra (2.4). Also note that by taking suitable combinations of the operators (2.2) one can move up and down in all possible ways the parameters n, α and β characterizing the basis functions $\{f_n^{(\alpha,\beta)}\}$. In view of this, the algebra (2.4) can be called the symmetry algebra of the Al-Salam–Chihara polynomials. In other words, the set of relations (2.8) constitutes a complete description of these polynomials.

3. Expansion formula

Many properties that the Al-Salam–Chihara polynomials satisfy can be obtained in a purely algebraic way using the symmetry model (2.8). As an example, we shall now derive an expansion formula for these polynomials involving a q-generalization of the exponential function. Various q-exponential functions have been introduced in the literature [1, 15, 16]. The one that is naturally connected with the families of continuous q-orthogonal polynomials is the eigenfunction of the divided-difference operator τ in (2.1a). In its most general form, it depends on two variables, $x = \frac{1}{2}(z + 1/z)$, $z = \mathrm{e}^{\mathrm{i}\theta}$ and $y = \frac{1}{2}(w + 1/w)$, $w = \mathrm{e}^{\mathrm{i}\varphi}$, and on a parameter ω [16, 17]:

$$
\begin{aligned}
\mathcal{E}_q(x, y; \omega) = {}& \frac{(q^{1/2}\omega^2; q^2)_\infty}{(q^{3/2}\omega^2; q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{k(k+1)/4}}{(q; q)_k} (-1)^k\\
&\times\left(-zwq^{(1-k)/2}, -\frac{w}{z}q^{(1-k)/2}; q\right)_k \left(\frac{\omega}{w}\right)^k.
\end{aligned}
\tag{3.1}
$$

This function satisfies the following characteristic properties [16]:

$$\mathcal{E}_q(x, y; \omega) = \mathcal{E}_q(y, x; \omega) \tag{3.2a}$$

$$\tau_x \mathcal{E}_q(x, y; \omega) = \omega \mathcal{E}_q(x, y; \omega) \tag{3.2b}$$

$$\mathcal{E}_q(0, 0; \omega) = 1 \tag{3.2c}$$

$$\mathcal{E}_q(x, y; \omega) = \mathcal{E}_q(x, 0; \omega)\mathcal{E}_q(0, y; \omega). \tag{3.2d}$$

(For the sake of clarity, in (3.2b) we have written τ_x for the operator defined in (2.1a) to explicitly indicate that it acts on the variable x; due to (3.2a), a similar relation holds for τ_y.) While the first three properties above are a direct consequence of identities satisfied by the q-shifted factorials, the last one is non-trivial and can be given an algebraic interpretation.

This interpretation is based on the symmetry algebra of the q-Hermite polynomials (1.7), the zero-parameter limit of the Al-Salam–Chihara polynomials. In this limiting case, the algebra (2.4) essentially reduces to the q-Euclidean algebra generated by the operators A_+, B_+ and P acting on the basis functions $f_n(x) = H_n(x|q)$. The function $\mathcal{E}_q(x, y; \omega)$, i.e. the q-exponential of the operator $P/2 \equiv x$, depending parametrically on y and ω, can be expressed in terms of the basis functions $\{f_n(x)\}$:

$$\mathcal{E}_q(x, y; \omega) = \sum_{k=0}^{\infty} U_k(y; \omega) f_k(x). \tag{3.3}$$

Indeed, $\mathcal{E}_q(x, y; \omega)$ can be thought of as acting on the vector $f_0 \equiv 1$. Applying the operator $A_+ = \tau_x/(q-1)$ to both sides of this equation, one obtains the following recurrence relation for the coefficients U_k:

$$\omega U_k(y; \omega) = -q^{-(k+1)/2}(1 - q^{k+1})\, U_{k+1}(y; \omega). \tag{3.4}$$

The dependence on the index k is then fixed, up to an arbitrary function,

$$U_k(y; \omega) = \frac{q^{k(k+1)/4}}{(q; q)_k}(-1)^k \omega^k \; U_0(y; \omega) \tag{3.5}$$

so that

$$\mathcal{E}_q(x, y; \omega) = U_0(y; \omega) \sum_{k=0}^{\infty} \frac{q^{k(k+1)/4}}{(q; q)_k}(-1)^k \omega^k \; H_k(x|q). \tag{3.6}$$

Setting $x = 0$ allows one to determine the function U_0: $\mathcal{E}_q(0, y; \omega) = U_0(y; \omega)(q^{3/2}\omega^2; q^2)_\infty$. Substituting this result back into (3.6) and recalling (3.2c), one first finds for $y = 0$ the expansion formula [8, 17]:

$$\mathcal{E}_q(x, 0; \omega) = \frac{1}{(q^{3/2}\omega^2; q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{k(k+1)/4}}{(q; q)_k}(-1)^k \omega^k \; H_k(x|q) \tag{3.7}$$

and hence the identity (3.2d).

An expansion formula similar to (3.7) also holds for the Al-Salam–Chihara polynomials, and it can be derived using model (2.8) for the q-algebra $\mathcal{G}_q$. One starts by observing that the operators K_1 and K_2 are diagonal on the basis $\{f_n^{(\alpha,\beta)}\}$ and that both commute with P. Therefore, the q-exponential $\mathcal{E}_q(x; \omega) \equiv \mathcal{E}_q(x, 0; \omega)$ of the generator $P/2$ acting on the function $f_0^{(\alpha,\beta)}(x, t, s) = t^\alpha s^\beta$ can be expressed as

$$\mathcal{E}_q(x; \omega) f_0^{(\alpha,\beta)}(x, t, s) = \sum_{n=0}^{\infty} W_n^{(\alpha,\beta)}(\omega) f_n^{(\alpha,\beta)}(x, t, s). \tag{3.8}$$

No summation over α and β occurs because the left-hand side is an eigenfunction of K_1 and K_2, and the same must be true for the right-hand side.

The matrix elements $W_n^{(\alpha,\beta)}(\omega)$ can be determined by solving the following recurrence relations,

$$\omega W_n^{(\alpha+1,\beta+1)}(\omega) = -q^{-(n+1)/2}(1 - q^{n+1}) W_{n+1}^{(\alpha,\beta)}(\omega) \tag{3.9a}$$

$$W_n^{(\alpha+2,\beta)}(\omega) = W_n^{(\alpha,\beta)}(\omega) - q^{\alpha/2}(1 - q^{n+1}) W_{n+1}^{(\alpha,\beta)}(\omega) \tag{3.9b}$$

$$W_n^{(\alpha,\beta+2)}(\omega) = W_n^{(\alpha,\beta)}(\omega) - q^{\beta/2}(1 - q^{n+1}) W_{n+1}^{(\alpha,\beta)}(\omega) \tag{3.9c}$$

that are obtained by acting with the operators A_+, Q_1 and Q_2 on both sides of (3.8), and using (2.8).

In solving these relations it proves convenient to rewrite $W_n^{(\alpha,\beta)}(\omega)$ in the following product form:

$$W_n^{(\alpha,\beta)}(\omega) = w_n(\omega) g(q^{(\alpha+n)/2}, q^{(\beta+n)/2}; \omega). \tag{3.10}$$

As a consequence, (3.9a) reduces to (3.4),

$$w_{n+1}(\omega) = -\frac{q^{(n+1)/2}\omega}{1-q^{n+1}} w_n(\omega) \tag{3.11}$$

which fixes $w_n(\omega)$ up to a function $w_0(\omega)$:

$$w_n(\omega) = \frac{q^{n(n+1)/4}}{(q;q)_n}(-1)^n \omega^n \ w_0(\omega). \tag{3.12}$$

Introducing for simplicity the new variables $u = q^{(\alpha+n)/2}$ and $v = q^{(\beta+n)/2}$, n fixed, the remaining two conditions in (3.9) become

$$(D_u + q^{1/2}\omega T_u^{1/2} T_v^{1/2}) g(u, v; \omega) = 0 \tag{3.13a}$$

$$(D_v + q^{1/2}\omega T_u^{1/2} T_v^{1/2}) g(u, v; \omega) = 0 \tag{3.13b}$$

where $D_z \equiv (1/z)(1 - T_z)$ is another standard definition of a q-derivative. The solution of (3.13) can be expressed in terms of Carlitz's q-Hermite polynomials [3, 18]:

$$h_n(z; q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} z^k. \tag{3.14}$$

These polynomials are orthogonal on the unit circle, $z = \mathrm{e}^{\mathrm{i}\theta}$, and satisfy the condition $D_z h(z; q) = (1 - q^n) h_{n-1}(z; q)$. Using this property, one can check that

$$g(u, v; \omega) = \sum_{k=0}^{\infty} \frac{q^{k(k+1)/4}}{(q;q)_k} (-1)^k h_k(u/v)(v\omega)^k \tag{3.15}$$

solves (3.13). (An arbitrary function of ω should also appear in the solution (3.15); it has been absorbed in $w_0(\omega)$.) The function g in (3.15) can also be expressed in terms of the q-exponential $\mathcal{E}_q$; indeed, it is not hard to show that

$$g(u, v; \omega) = (q^{3/2} uv\omega^2; q^2)_\infty \, \mathcal{E}_q[\tfrac{1}{2}((u/v)^{1/2} + (v/u)^{1/2}); \omega\sqrt{uv}]. \tag{3.16}$$

It remains to fix the function $w_0(\omega)$. This can be easily done by letting $q^\alpha = q^\beta = 0$, so that (3.8) must coincide with the expansion (3.7) for the continuous q-Hermite polynomials. (This condition has already been implicitly used in the choice of the solution in (3.15).) In this way, one obtains $w_0(\omega) = 1/(q^{3/2}\omega^2; q^2)_\infty$. Putting all the pieces together, one finally obtains the following expansion formula:

$$\mathcal{E}_q(x; \omega) = \frac{1}{(q^{3/2}\omega^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4}}{(q;q)_n} (-1)^n \omega^n \times g(q^{(\alpha+n)/2}, q^{(\beta+n)/2}; \omega) Q_n(x; q^{\alpha/2}, q^{\beta/2} | q). \tag{3.17}$$

This is the analogue for the Al-Salam–Chihara polynomials of the Fourier–Gegenbauer relation that involves the expansion of a plane wave in terms of Jacobi polynomials [11, 13, 19]. Note that when either q^α or q^β vanishes, (3.17) reduces to the expansion formula for the continuous big q-Hermite polynomials presented in [9].

The constructive derivation of the identity (3.17) using symmetry techniques clearly illustrates the power of the algebraic approach to the theory of q-orthogonal polynomials.

It is of interest to study the properties of other families of q-continuous polynomials in an analogous fashion, and in particular to determine the symmetry algebra of the Askey–Wilson polynomials. Work along these lines is in progress.

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